

QUADROTOR ROBUST PATH TRACKING WITHOUT VELOCITY MEASUREMENTS USING THE GENERALIZED SUPER-TWISTING CONTROL

SEGUIMIENTO DE TRAYECTORIA ROBUSTA DE UN CUADRICÓPTERO SIN MEDICIONES DE VELOCIDAD UTILIZANDO EL CONTROL SUPER-TWISTING GENERALIZADO

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ABSTRACT

This paper presents a nonlinear control strategy to solve the path tracking problem for a quadrotor unmanned aerial vehicle under perturbations. This strategy is based on the Generalized Super-Twisting Algorithm (GSTA); it means the second order sliding mode technique, which is able to ensure robustness with respect to modeling errors and bounded external disturbances due to the added extra linear correction terms. The controller goal is to achieve suitable path tracking of desired absolute positions and yaw angle while keeping the stability of the pitch and roll angle, in spite of the presence of disturbances and the handling of all system nonlinearities. In this work, a scenario in which velocities measurements are not available and are estimated by the Generalized Super-Twisting Observer is considered. Finally, the simulation results are also provided in order to illustrate the performances of the proposed controller.

Keywords: Quadrotor; Generalized Super Twisting; Sliding mode control; Observer; non-linear control.

Received: June 16, 2017.

Recibido: 16 de Junio de 2017.

Accepted: September 08th, 2017.

Aceptado: 08 de Septiembre de 2017.

RESUMEN

Este artículo presenta una estrategia de control no lineal para resolver el problema de seguimiento de trayectorias para un vehículo aéreo no tripulado bajo perturbaciones. Esta estrategia se basa en el Algoritmo Super-Twisting Generalizado (GSTA); es una técnica de modos deslizantes de segundo orden, la cual es capaz de asegurar robustez con respecto a errores de modelado y perturbaciones externas acotadas debido a los términos de corrección lineales añadidos respecto al algoritmo Super Twisting convencional. El objetivo del controlador es conseguir un seguimiento de trayectoria adecuado de las posiciones absolutas deseadas y del ángulo de guiñada, mientras se mantiene la estabilidad del ángulo de inclinación y de alabeo, a pesar de la presencia de perturbaciones y las no linealidades del sistema. En este trabajo, es considerado un escenario en el que las mediciones de las velocidades no están disponibles y son estimadas por el Observador Super-Twisting Generalizado. Finalmente, también fueron proporcionados los resultados de simulación para ilustrar el desempeño del controlador propuesto.

Palabras clave: Cuadricóptero; Super-Twisting Generalizado; control por modo deslizante; Observador; control no lineal.

Cómo citar este artículo: J. N. Guerrero-Tavares, S. Ortiz-Santos, J. L. Ortiz-Solano, "Quadrotor robust path tracking without velocity measurements using the generalized super-twisting control," *Revista Politécnica*, vol. 13, no. 25, pp. 115-124, 2017.

1. INTRODUCTION

Unmanned Aerial Vehicles (UAVs) constitute a field that has motivated the control community during the last years. UAVs are used in many applications such as, search, rescue, surveillance, aerial cinematography, inspection, and mapping. This paper deals with a four-rotor helicopter known as quadrotor, shown in Figure 1. The used quadrotor helicopter has some advantages over conventional helicopters on inspection task. They can take off and land in limited spaces and hover. The control of this machine is a great challenge, therefore, it is suited for the development and testing of new control strategies due to its high nonlinearity of its dynamic model [1]. Moreover, this system is under-actuated because it has six degrees of freedom while it has only four inputs (four rotors). It is not possible to control all of the states at the same time. On the other side, several controllers have been developed to stabilize the attitude of quadrotors, such as, the feedback linearization method [2] or nonlinear control techniques like backstepping [1], [3],[4],[5] and [6], sliding-mode techniques [1], [3], H_∞ controller [7], sliding mode control driven by sliding mode disturbance observer [8], adaptive trajectory tracking control algorithm [9], controller parameterized [10], super twisting control algorithm for trajectory tracking [11], [12], nonlinear adaptive state feedback controller [13]. This paper deals with the design of a nonlinear control algorithm for the position and attitude tracking of the quadrotor taking into account nonholonomic constraints. Added to this, it is considered the case in which velocity measurements are not available, then an observer is designed to estimate these variables based in GSTA. The GSTA is able to ensure robustness with respect to bounded external disturbances while reducing the chattering phenomenon caused by all first order sliding mode based controllers. This technique differs from conventional Super-Twisting Algorithm due to the introduction of extras linear terms, that provide more robustness and convergence velocity [14].

The paper is outlined as follows. In section II, the dynamic model of the quadrotor UAV is developed. Based on this nonlinear model, we design in section III, Super-Twisting Algorithm controller. In section IV an observer to estimate the velocities of the quadrotor is designed. Some simulation results are carried out, allowing the analysis of the stability, robustness of the proposed controller in the presence of perturbations are presented in section

V. Finally, section VI is a conclusion. The main contributions of this article are described as follows: Design of an attitude and position controller based on the Generalized Super-Twisting Controller. Design of a state observer based on GSTA for robust velocity estimation. Alternative stability proof for the GSTA.

2. QUADROTOR MODEL

The dynamic model helicopter has been described in several articles as seen in [1], [15] [16], [17], [18] and [19]. The absolute position and the attitude of the quadrotor are described respectively by $X = [x, y, z]^T$ and the Euler angles $\theta = [\psi, \theta, \phi]^T$, corresponding to an aeronautical convention. The equation of the dynamics of rotation of the quadrotor will be:

$$\ddot{X} = \frac{1}{m} R(\theta) [F_{prop} - A_F(U)] + g e_3 \quad (1)$$

$$\ddot{\theta} = (JM(\theta))^{-1} [T_{prop} - JN(\theta, \dot{\theta}) - A_T(U) - (M(\theta)\dot{\theta}) \times (JM(\theta)\dot{\theta})] \quad (2)$$

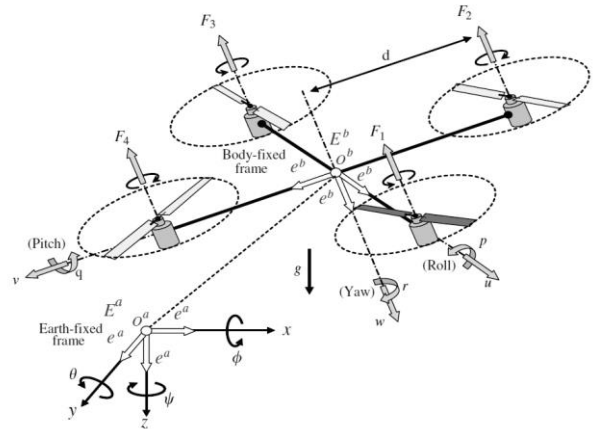


Fig. 1. Quadrotor schematic taken from [11].

Where $M(\theta)$ and $N(\theta, \dot{\theta})$ are given by:

$$M(\theta) = \begin{bmatrix} S_\theta & 0 & 1 \\ C_\theta S_\phi & C_\phi & 0 \\ C_\theta C_\phi & -S_\phi & 0 \end{bmatrix}$$

$$N(\theta, \dot{\theta}) = \begin{bmatrix} -C_\theta \dot{\theta} \dot{\psi} \\ -S_\phi \dot{\phi} \dot{\theta} + C_\phi C_\theta \dot{\phi} \dot{\psi} - S_\phi S_\theta \dot{\theta} \dot{\psi} \\ -C_\phi \dot{\phi} \dot{\theta} - S_\phi C_\theta \dot{\phi} \dot{\psi} - C_\phi S_\theta \dot{\theta} \dot{\psi} \end{bmatrix}$$

The orthogonal rotation matrix $R(\Theta)$ is defined as follows:

$$R(\Theta) = \begin{bmatrix} C_\psi C_\theta & C_\psi S_\theta S_\phi - S_\psi C_\phi & C_\psi S_\theta C_\phi + S_\psi S_\phi \\ S_\psi C_\theta & S_\psi S_\theta S_\phi - C_\psi C_\phi & S_\psi S_\theta C_\phi - C_\psi S_\phi \\ -S_\theta & C_\theta S_\phi & C_\theta C_\phi \end{bmatrix}$$

Where $\cos(*) = C_*$ and $\sin(*) = S_*$. The inertia matrix of the quadrotor is represented as $J = \text{diag}[I_x, I_y, I_z]$. g is the gravity; the vector $e_3 = [0; 0; 1]^T$; m is the mass of the vehicle; U is the velocity of the quadrotor with respect to the air; $A_F(U)$ and $A_T(U)$ are respectively two complex non-linear functions vectors which represent aerodynamic forces and torques in the propeller system. Finally, F_{prop} and T_{prop} are the force and torque vectors in the system given by:

$$F_{prop} = \sum_{i=1}^4 F_i e_3$$

$$T_{prop} = \begin{bmatrix} d(F_2 - F_4) \\ d(F_1 - F_3) \\ c(F_1 - F_2 + F_3 - F_4) \end{bmatrix}$$

Where $c > 0$ is the drag factor.

3. CONTROLLER DESIGN

In this section, we design the controllers for the position and attitude of quadrotor using the Super-Twisting Algorithm.

3.1 Attitude controller design

From (2), we define the following state variables:

$$\dot{q}_1 = \theta \quad \dot{q}_2 = \dot{\theta}$$

Then the state-space form of this model is given by:

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= f(q) + p(q)u + w(t) \end{aligned} \quad (3)$$

Where $w(t) = [w_1; w_2; w_3]^T$ is the vector of disturbance; $T_{prop} = u = [u; u; u]^T$ is the vector of attitude control. The vectors $f(q)$ and $p(q)$ are defined as follows:

$$f(q) = -(JM(\theta))^{-1} [JN(\theta, \dot{\theta}) + (M(\theta)\dot{\theta}) \times (JM(\theta)\dot{\theta})] \quad (4)$$

$$p(q) = (JM(\theta))^{-1} \quad (5)$$

For the design of control, the following assumptions are needed [1]:

Assumption 1. $X, \theta, \dot{\theta}$ can be measured or

estimated by on-board sensors.

Assumption 2. The quantity $\sum_{i=1}^4$ represents the total thrust on the body in the z axis and is strictly positive to overcome the gravity.

Assumption 3. The pitch, roll and yaw angles are limited to $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$ and $-\pi < \psi < \pi$.

Assumption 4. The desired trajectories and their first and second time derivatives are bounded.

Assumption 5. The velocity U and \dot{U} the acceleration of the quadrotor with respect to the air are bounded.

According to these assumptions, the matrix (5) is non-singular and the following inequalities are satisfied:

$$w_i(t) \leq \delta_i, \quad i = 1, 2, 3.$$

The controller is designed to enforce sliding mode [11] on the manifold:

$$\sigma = \dot{e} + \Gamma e \quad (6)$$

Where $e = q_1^d - q_1$, $\dot{e} = \dot{q}_1^d - \dot{q}_1$ with $q_1^d(t) = [\varphi_d(t), \theta_d(t), \psi_d(t)]^T$ and $\Gamma \in \mathbb{R}^{3 \times 3}$ is a diagonal positive definite matrix

The proposed control law is in the form:

$$u = -p(q)^{-1}(v + \ddot{q}_1^d - \Gamma \dot{e} - f(q)) \quad (7)$$

Where v is the Super-Twisting sliding-mode control given by:

$$v = -K_1 \phi_1(\sigma) + \lambda \quad (8)$$

$$\dot{\lambda} = -K_2 \phi_2(\tau) \quad (9)$$

with:

$$\begin{aligned} \phi_1(\sigma) &= \sigma^{\frac{1}{2}} \text{sgn}(\sigma) + \sigma \\ \phi_2(\sigma) &= \frac{1}{2} \text{sgn}(\sigma) + \frac{3}{2} \sigma^{\frac{1}{2}} \text{sgn}(\sigma) + \sigma \end{aligned} \quad (10)$$

Where $\text{sgn}(\cdot)$ denotes the sign function and is defined as:

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \in [-1, 1] & \text{if } x = 0 \end{cases}$$

$K_1 = \text{diag}[k_{11}, k_{12}, k_{13}]$ and $K_2 = \text{diag}[k_{21}, k_{22}, k_{23}]$ are the gain positive definite matrices.

3.2 Position controller design

From (1), we have the following state variables:

$$\dot{x} = \left(\frac{h_1}{m} \right) (F_{prop} - A_F(U)) \quad (11)$$

$$\begin{aligned}\ddot{y} &= \left(\frac{h_2}{m}\right) (F_{prop} - A_F(U)) \\ \ddot{z} &= \left(\frac{h_3}{m}\right) (F_{prop} - A_F(U)) - g\end{aligned}$$

Where:

$$\begin{aligned}h_1 &= S_\theta C_\psi C_\phi + S_\psi S_\phi \\ h_2 &= S_\theta S_\psi C_\phi - C_\psi S_\phi \\ h_3 &= C_\theta C_\phi\end{aligned}$$

Let the state variables defined by:

$$\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \\ z \\ \dot{z} \end{bmatrix}$$

Now, the system (11) is given in the next form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \left(\frac{h_1}{m}\right) (F_{prop} - A_F(U)) \\ y_2 \\ \left(\frac{h_2}{m}\right) (F_{prop} - A_F(U)) \\ z_2 \\ -g + \left(\frac{h_3}{m}\right) (F_{prop} - A_F(U)) \end{bmatrix} \quad (11)$$

Following the methodology of the previous section, the controller is designed to enforce sliding mode on the manifold:

$$\zeta = \dot{r} + \Delta r \quad (13)$$

Where the position error and its derivative are defined respectively as $r = [x_1^d - x_1, y_1^d - y_1, z_1^d - z_1]^T$ and $\dot{r} = [\dot{x}_1^d - \dot{x}_1, \dot{y}_1^d - \dot{y}_1, \dot{z}_1^d - \dot{z}_1]^T$ with $\Delta \in \mathbb{R}^{3 \times 3}$ is a diagonal positive definite matrix. $A_F(U)$ is considered as a bounded perturbation.

The Super-Twisting controller:

$$\begin{aligned}v_i &= -K_3 \phi_3(\zeta) + \alpha_i \\ \dot{\alpha}_i &= -K_4 \phi_4(\zeta)\end{aligned} \quad (14)$$

With $i = 1, 2, 3$. Where:

$$\begin{aligned}\phi_3(\zeta) &= \zeta^{\frac{1}{2}} \text{sgn}(\zeta) + \zeta \\ \phi_4(\zeta) &= \frac{1}{2} \text{sgn}(\zeta) + \frac{3}{2} \zeta^{\frac{1}{2}} \text{sgn}(\zeta) + \zeta\end{aligned} \quad (15)$$

The gain matrices $K_3 = \text{diag}[k_{31}, k_{32}, k_{33}]$, $K_4 = \text{diag}[k_{41}, k_{42}, k_{43}]$ are positive definite.

From (12) the controller for z is obtained as:

$$U_z = F_{prop} = \left(\frac{m}{h_3}\right) (g + v_3) \quad (16)$$

Note from (12) that is only possible to control the attitude of quadrotor directly due to the form of F_{prop} , while the $(x - y)$ control is done by modifying the Euler angles $(\phi - \theta)$. Therefore, a first step in controlling the position of the quadrotor on the $(x - y)$ axes, a virtual controller is proposed as follows:

$$U_x = h_1 = \left(\frac{m}{U_z}\right) v_1 \quad (17)$$

$$U_y = h_2 = \left(\frac{m}{U_z}\right) v_2 \quad (18)$$

From the equations above, U_z only tends to zero when the helicopter is on the ground, therefore, the expression (17)-(18) is not indeterminate.

The next step is to find algebraic relations between virtual controllers and the Euler angles desired. From (11) without considering the effect of the given disturbance and maintaining the variable ψ_d free, we obtained the following expressions:

$$\tan \theta_d = \frac{\ddot{x} \cos \psi_d + \ddot{y} \sin \psi_d}{\ddot{z} + g} \quad (19)$$

$$\sin \phi_d = \frac{m(\ddot{x} \cos \psi_d + \ddot{y} \sin \psi_d)}{\sqrt{\ddot{x}^2 + \ddot{y}^2 + (\ddot{z} + g)^2}} \quad (20)$$

4. OBSERVER DESIGN

In this section, the design of a state observer for estimating speed variables is shown. The high-order sliding mode observer to estimate the altitude velocity has already been implemented [20]. We propose a scenario where only the position variables (x, y, z) are measured and their speed variables are not available, under this assumption is possible to recover exactly and in time finite with only the position measurement using the Generalized Super-Twisting observer [14]:

$$\begin{aligned}\hat{\dot{x}}_1 &= -\kappa_1 \bar{\phi}_1(\hat{x}_1 - x) + \hat{x}_2 \\ \hat{\dot{x}}_2 &= -\kappa_2 \bar{\phi}_2(\hat{x}_1 - x) \\ \hat{\dot{y}}_1 &= -\kappa_3 \bar{\phi}_3(\hat{y}_1 - x) + \hat{y}_2 \\ \hat{\dot{y}}_2 &= -\kappa_4 \bar{\phi}_4(\hat{y}_1 - x) \\ \hat{\dot{z}}_1 &= -\kappa_5 \bar{\phi}_5(\hat{z}_1 - x) + \hat{z}_2 \\ \hat{\dot{z}}_2 &= -\kappa_6 \bar{\phi}_6(\hat{z}_1 - x)\end{aligned} \quad (21)$$

Where κ_i with $i = \overline{1,6}$ are the gains of the algorithm and

$$\begin{aligned}\bar{\phi}_{2n+1}(\chi_n) &= \chi_n^{\frac{1}{2}} \text{sgn}(\chi_n) + \chi_n \\ \bar{\phi}_{2n+2}(\chi_n) &= \frac{1}{2} \text{sgn}(\chi_n) + \frac{3}{2} \chi_n^{\frac{1}{2}} \text{sgn}(\chi_n) + \chi_n\end{aligned}$$

With $n = 0,1,2$ and $\chi = [\chi_0, \chi_1, \chi_2] = [\hat{x} - x, \hat{y} - y, \hat{z} - z]$ and $(\hat{x}, \hat{y}, \hat{z})$ are the estimated variables. Note that the use of discontinuous injection in $\bar{\phi}$ allows obtaining an exact estimate as discussed below.

5. SIMULATION RESULTS

In this section, we test the robustness of the proposed controller. The quadrotor model parameters used are: $m = 0.468 \text{ kg}$, $I_x = I_y = 3.83 \times 10^{-3} \text{ kg} \cdot \text{m}^2$, $I_z = 7.13 \times 10^{-3} \text{ kg} \cdot \text{m}^2$ and $g = 9.81 \text{ m/s}^2$.

The controller parameters for the attitude control were selected as: $\Gamma = \text{diag}[2,2,2]$, $K_1 = 4\sqrt{W}$, $K_2 = 2W$, where $W = \text{diag}[12,12,8]$. For the position controller: $\Delta = \Gamma$, $K_3 = 8\sqrt{W}$, $K_4 = 4W$.

For the observer, selected gains were as follows: $\kappa_1 = \kappa_3 = \kappa_5 = 10$ and $\kappa_2 = \kappa_4 = \kappa_6 = 2\kappa_1$. The proposed trajectory is a spiral whose three-dimensional parametric expression is given by the function:

$$\begin{bmatrix} x_d \\ y_d \\ z_d \end{bmatrix} = \begin{bmatrix} \sin(2t) \\ \cos(2t) - 1 \\ t \end{bmatrix} \quad (22)$$

The external disturbance in (1) is defined by $A_F(U) = 7H(x)$

Where:

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

This external disturbance lasted two seconds in the span of 2 to 3 seconds as shown in Figure 2. A low pass filter with transfer function, $LPF = 10/(s + 10)$ (s is the Laplace variable) is applied to ψ_d and θ_d in order to minimize the chattering effect in the references. The results show that the vehicle follows the desired path even in the presence of the large external perturbation.

The controller performance for the position and

orientation of the quadrotor respectively are shown in figures 2 and 3.

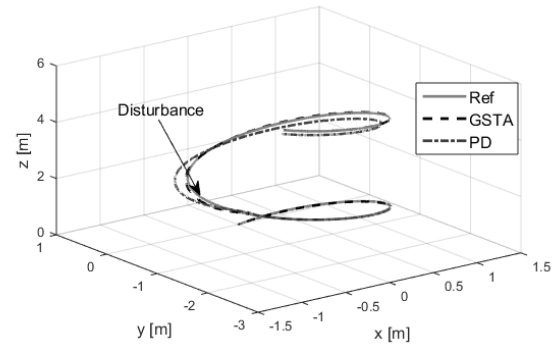


Fig. 2. Path tracking of the quadrotor.

Figures 3 and 4 show the orientation and the desired orientation of the quadrotor during the test maintaining $\psi_d = 0$.

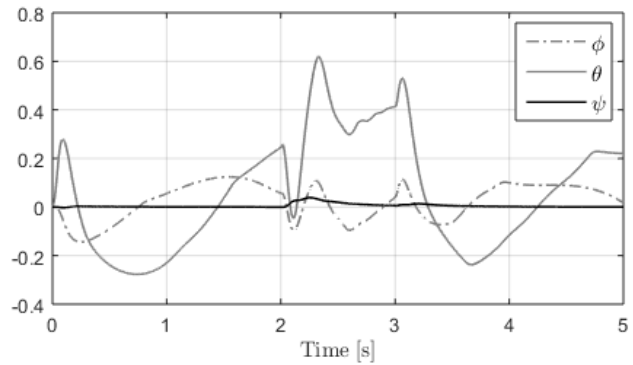


Fig. 3. Euler angles ϕ , θ and ψ in tracking disturbance test.

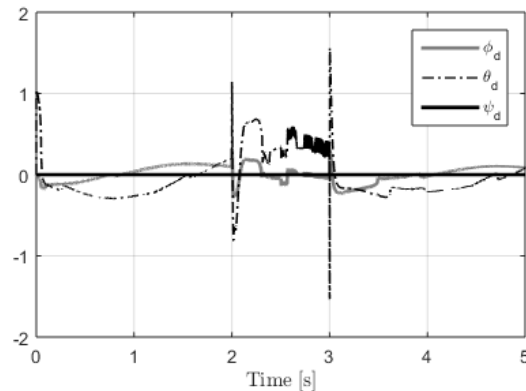


Fig. 4. Desired Euler angles ϕ_d , θ_d , ψ_d in tracking disturbance test.

Figures 5 and 6. Also, the effect of the external disturbances is shown. From these figures we can see that the control signals have low amplitude chattering.

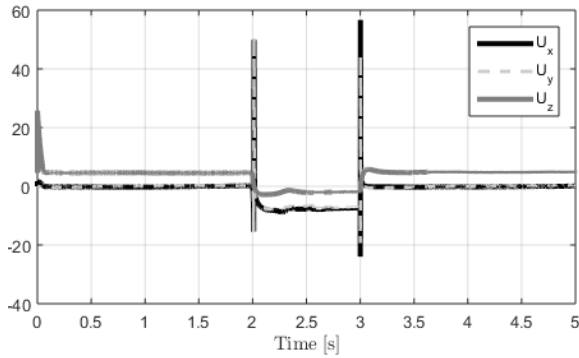


Fig. 5. Controller outputs in tracking disturbance test.

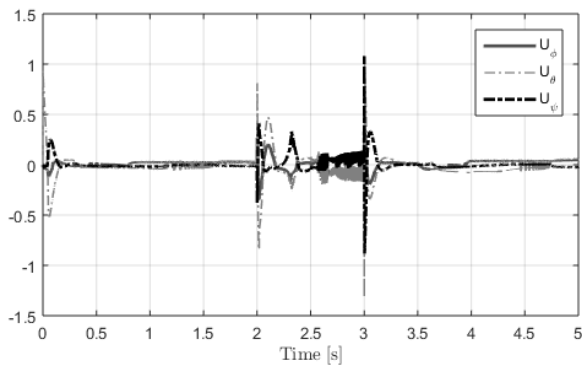


Fig. 6. Controller outputs in tracking disturbance test.

The Figure 7 shown the tracking error of the position controller. Also, these plots demonstrate that the position and the attitude of quadrotor are clearly affected by the external disturbance but the controller estimate the disturbance and compensate for it quickly such that the errors converge to zero.

Finally, Figure 8 shows the velocity estimation. From this plot we can see that the estimated velocities converge to the actual values despite the disturbance introduced in the test.

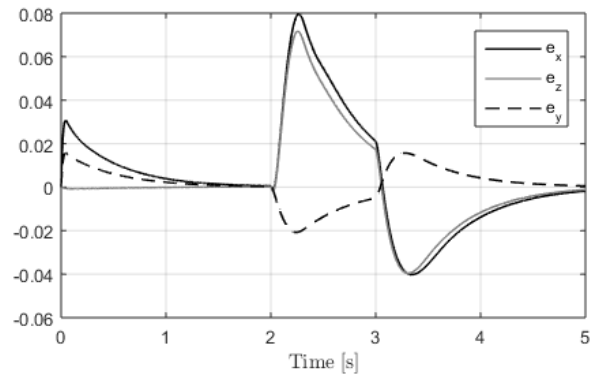


Fig. 7. Error tracking in disturbance test.

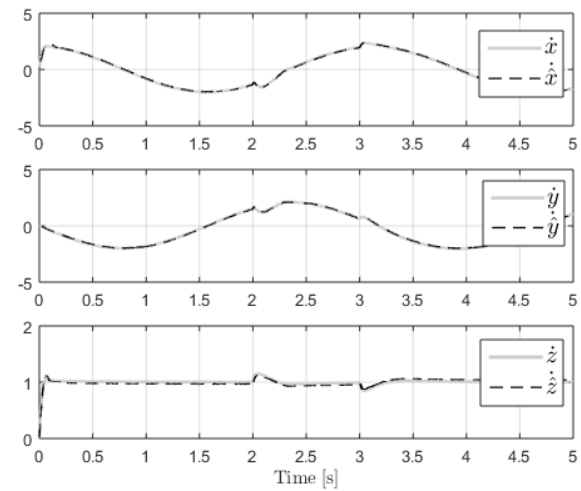


Fig. 8. Velocity variables (gray) and estimated velocity variables (dotted line) under perturbation test (2-3 seconds).

We show that the performance of the GSTA controller presents better results compared to a classic controller such as the PD controller (figure 9). To evaluate the performance of the controllers we use the average Euclidean distance, given by the following equation.

$$E_d = \frac{1}{N} \sum_{i=0}^N \sqrt{(x - x^*)^2 + (y - y^*)^2 + (z - z^*)^2}$$

Where x_d , y_d and z_d are the desired path, x^* , y^* and z^* are the trajectories obtained with the controllers. N is the number of samples.

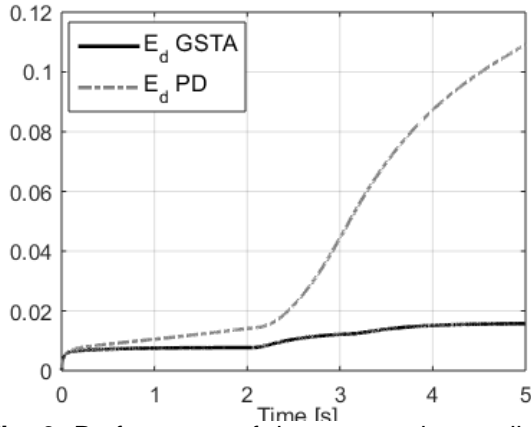


Fig. 9. Performance of the proposed controller and PD.

6. CONCLUSIONS

In this paper, a super twisting controller algorithm to solve the path tracking problem for a quadrotor has been presented. The controller has been designed using second order sliding mode approach in order to reduce chattering phenomenon and to ensure robustness with respect to external disturbances. The whole system was divided into six subsystems. Three of them represent the attitude of quadrotor and have implemented the super-twisting algorithm to stabilize the roll, pitch and yaw angles. The rest subsystems have implemented the super twisting algorithm too, in order to tracking control of quadrotor and taking into account nonholonomic constraints, it was possible modified the yaw and roll angles necessary for displacement on x and y axes. Also, it is assumed that the velocities are not available and were estimated by the Generalized Super-Twisting observer. The simulation results obtained on a quadrotor system show the effectiveness of the proposed controller in the stabilization, path tracking and disturbance rejection cases.

APPENDIX 1: LYAPUNOV STABILITY OF THE SUPER-TWISTING ALGORITHM

Based in [11]. From (6) and (11), the closed loop dynamics is given by:

$$\begin{aligned} \dot{\sigma} &= \omega(t) - K_1 \left[|\sigma|^{\frac{1}{2}} \text{sgn}(\sigma) + \sigma \right] \\ &\quad - K_2 \int_0^t \left[\frac{1}{2} \text{sgn}(\tau) + \frac{3}{2} |\tau|^{\frac{1}{2}} \text{sgn}(\tau) + \tau \right] d\tau \end{aligned} \quad (23)$$

$$\begin{aligned} \dot{\varsigma} &= \varpi(t) - K_3 \left[|\varsigma|^{\frac{1}{2}} \text{sgn}(\varsigma) + \varsigma \right] \\ &\quad - K_4 \int_0^t \left[\frac{1}{2} \text{sgn}(\tau) + \frac{3}{2} |\tau|^{\frac{1}{2}} \text{sgn}(\tau) + \tau \right] d\tau \end{aligned} \quad (24)$$

Let:

$$\begin{aligned} \mu_{1i} &= \sigma_i \\ \mu_{2i} &= -K_{2i} \int_0^t \left[\frac{1}{2} \text{sgn}(\tau) + \frac{3}{2} |\tau|^{\frac{1}{2}} \text{sgn}(\tau) + \tau \right] d\tau \\ \dot{\omega}(t) &= \beta_i(t) \\ \mu_{3j} &= \varsigma_j \\ \mu_{4j} &= -K_{4j} \int_0^t \left[\frac{1}{2} \text{sgn}(\tau) + \frac{3}{2} |\tau|^{\frac{1}{2}} \text{sgn}(\tau) + \tau \right] d\tau \\ \dot{\omega}(t) &= \gamma_j(t) \end{aligned}$$

Then (23) and (24) can be rewritten in scalar form ($i, j = \overline{1,3}$) as:

$$\begin{aligned} \dot{\mu}_{1i} &= -K_{1i} \left[|\mu_{1i}|^{\frac{1}{2}} \text{sgn}(\mu_{1i}) + \mu_{1i} \right] + \mu_{2i} \\ \dot{\mu}_{2i} &= -K_{2i} \left[\frac{1}{2} \text{sgn}(\mu_{1i}) + \frac{2}{3} |\mu_{1i}|^{\frac{1}{2}} \text{sgn}(\mu_{1i}) + \mu_{1i} \right] \\ &\quad + \beta_i(t) \\ \dot{\mu}_{3i} &= -K_{3j} \left[|\mu_{3j}|^{\frac{1}{2}} \text{sgn}(\mu_{3j}) + \mu_{3j} \right] + \mu_{4j} \\ \dot{\mu}_{4i} &= -K_{4j} \left[\frac{1}{2} \text{sgn}(\mu_{3j}) + \frac{2}{3} |\mu_{3j}|^{\frac{1}{2}} + \mu_{3j} \right] + \gamma_j(t) \end{aligned}$$

Without loss of generality, we can choose a single component of the system above and simplifying the notation as follows:

$$\begin{aligned} \dot{\mu}_1 &= -\bar{k}_1 \left[|\mu_1|^{\frac{1}{2}} \text{sgn}(\mu_1) + \mu_1 \right] + \mu_2 \\ \dot{\mu}_2 &= -\bar{k}_2 \left[\frac{1}{2} \text{sgn}(\mu_1) + \frac{2}{3} |\mu_1|^{\frac{1}{2}} \text{sgn}(\mu_1) \right. \\ &\quad \left. + \mu_1 \right] + \beta(t) \end{aligned} \quad (25)$$

The Lyapunov candidate function is given by

$$\begin{aligned} V(\mu) &= \frac{1}{2} \rho_2 (\bar{k}_1 |\mu_1|^{\frac{1}{2}} \text{sgn}(\mu_1) + \bar{k}_1 \mu_1 \\ &\quad - \mu_2^2)^2 + \frac{1}{2} \rho_1 \mu_2^2 + (\rho_1 \\ &\quad + \rho_2) \bar{k}_2 \left[|\mu_1| + \frac{1}{2} |\mu_1|^3 \right. \\ &\quad \left. + |\mu_1|^{\frac{3}{2}} \right] \end{aligned} \quad (26)$$

Note that the proposed Lyapunov candidate

function is a continuous, positive definite and differentiable function except in $\mu_1 = 0$. This function can be rewritten in quadratic form $V(\mu) = \xi^T P \xi$ where $\xi^T = [|\mu_1|^{\frac{1}{2}} \text{sgn}(\mu_1), \mu_1, \mu_2]$, then:

$$p = \frac{1}{2} \begin{bmatrix} \rho_1 \bar{k}_1^2 + 2L\bar{k}_2 & \rho_2 \bar{k}_1^2 + L\bar{k}_2 & -\rho_2 \bar{k}_1 \\ \rho_2 \bar{k}_1^2 + L\bar{k}_2 & \rho_2 \bar{k}_1^2 + L\bar{k}_2 & -\rho_2 \bar{k}_1 \\ -\rho_2 \bar{k}_1 & -\rho_2 \bar{k}_1 & L \end{bmatrix}$$

where $L = \rho_1 + \rho_2$. If the scalar free parameters $\rho_1, \rho_2, \bar{k}_1$ and \bar{k}_2 are chosen positive, then $V(\mu)$ is a positive definite matrix.

Now, this form satisfies:

$$\lambda_{\min}(P) \|\xi\|_2^2 \leq V(\mu) \leq \lambda_{\max}(P) \|\xi\|_2^2 \quad (27)$$

Where $\|\xi\|_2^2 = |\mu_1| + \mu_1^2 + \mu_2^2$ is the Euclidean norm of ξ . The time derivative along the trajectories of the system is:

$$\begin{aligned} \dot{V} \leq & -\xi^T [Q_1 + Q_2] \xi - \frac{1}{|\mu_1|^{\frac{1}{2}}} \xi^T [R_1 + R_2] \xi \\ & + \frac{1}{|\mu_1|^{\frac{1}{2}}} v_1^T \xi \end{aligned} \quad (28)$$

Where:

$$\begin{aligned} Q_1 &= \bar{k}_1 \begin{bmatrix} 2\rho_2 \bar{k}_1^2 & 0 & 0 \\ 0 & \rho_2 \bar{k}_1^2 & -\rho_2 \bar{k}_1 \\ 0 & -\rho_2 \bar{k}_1 & \frac{1}{2}\rho_2 \end{bmatrix} \\ Q_2 &= \bar{k}_1 \begin{bmatrix} \bar{k}_2 (\frac{5}{2}\rho_1 + \frac{1}{2}\rho_2) & 0 & 0 \\ 0 & \bar{k}_2 (2\rho_1 + \rho_2) & -\frac{1}{2}L \frac{\bar{k}_2}{\bar{k}_1} \\ 0 & -\frac{1}{2}L \frac{\bar{k}_2}{\bar{k}_1} & \frac{1}{2}\rho_2 \end{bmatrix} \\ R_1 &= \bar{k}_1 \begin{bmatrix} \rho_1 \bar{k}_1^2 & 0 & 0 \\ 0 & 5\rho_2 \bar{k}_1^2 & -3\rho_2 \bar{k}_1 \\ 0 & -3\rho_2 \bar{k}_1 & \frac{1}{2}\rho_2 \end{bmatrix} \\ R_2 &= \bar{k}_1 \begin{bmatrix} \bar{k}_2 (2\rho_1 + \rho_2) & 0 & -\frac{1}{2}L \frac{\bar{k}_2}{\bar{k}_1} \\ 0 & \bar{k}_2 (7\rho_1 + 2\rho_2) & 0 \\ -\frac{1}{2}L \frac{\bar{k}_2}{\bar{k}_1} & 0 & \frac{1}{2}\rho_2 \end{bmatrix} \\ v_1^T &= \delta [\rho_2 \bar{k}_1 \quad \rho_2 \bar{k}_1 \quad L] \end{aligned}$$

Suppose that the perturbation terms of (25) are globally bounded by Assumption 5. Then, the time derivative of the Lyapunov function can be rewritten as follows:

$$\dot{V}(\mu) \leq -\xi^T Q \xi - \frac{1}{|\mu_1|^{\frac{1}{2}}} \xi^T R \xi$$

Where $Q = Q_1 + Q_2$ and $R = R_1 + R_2 - \delta R_3$. The matrix R_3 is given by:

$$R_3 = \begin{bmatrix} \rho_2 \bar{k}_1 & \frac{1}{2}\rho_2 \bar{k}_1 & \frac{1}{2}L \\ \frac{1}{2}\rho_2 \bar{k}_1 & 0 & 0 \\ \frac{1}{2}L & 0 & 0 \end{bmatrix}$$

Choosing the parameters $\rho_1 > 0, \rho_2 > 0$ the gains $\bar{k}_1 > 0, \bar{k}_2 > \delta$ and also high enough, matrices Q and R are positive definite. Under the conditions described:

$$\dot{V}(\mu) \leq -\frac{1}{|\mu_1|^{\frac{1}{2}}} \xi^T R \xi \leq -\frac{1}{|\mu_1|^{\frac{1}{2}}} \xi^T \lambda_{\min}(R) \|\xi\|_2^2 \quad (29)$$

Using (27), (29) and also:

$$|\mu_1|^{\frac{1}{2}} \leq \|\xi\|_2 \geq \frac{V^{\frac{1}{2}}(\mu)}{\lambda_{\min}^{\frac{1}{2}}(P)} \quad (30)$$

It follows that:

$$\dot{V}(\mu) \leq -\epsilon V^{\frac{1}{2}}(s)$$

$$\text{With } \epsilon = \frac{\lambda_{\min}^{\frac{1}{2}}(P) \lambda_{\min}(Q_2)}{\lambda_{\max}(Q_2)}$$

Solving its analog differential equation:

$$\dot{v} = -\epsilon v^{\frac{1}{2}}, v(0) = v_0 \geq 0$$

The solution is given by:

$$v(t) = (v_0^{\frac{1}{2}} - \frac{\epsilon}{2} t)^2$$

And using the comparison principle, the solution converges in finite time to the origin with $T = 2V^{\frac{1}{2}}(\mu_0) \backslash \epsilon$ units of time.

Finally, $\mu_1 = 0$ in finite time and according to (6)

and (13) this implies that $\lim_{t \rightarrow \infty} e = 0$ and $\lim_{t \rightarrow \infty} \dot{e} = 0$
 and $\lim_{t \rightarrow \infty} r = 0$ and $\lim_{t \rightarrow \infty} \dot{r} = 0$

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